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# From stirring to mixing of momentum: cascades from balanced flows to dissipation in the oceanic interior

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**Abstract.** Under the influences of stable density stratification and Earth's rotation, large-scale flows in the ocean and atmosphere have a mainly balanced dynamics—sometimes called the slow manifold—in the sense that there are diagnostic hydrostatic and gradient-wind balances that constrain the fluid acceleration. The nonlinear balance equations are a successful approximate model for this regime, and we have identified mathematically explicit limits of their time integrability. We hypothesize that these limits are indicative, at least approximately, of the transition from the larger-scale regime of inverse energy cascades of anisotropic flows to the smaller-scale regimes of forward energy cascade to dissipation of more nearly isotropic flows and intermittently breaking inertia-gravity waves. In the oceans these regime transitions occur mostly in the scale range of 0.1-10 km—in between the mesoscale and fine-structure—where Rossby ( $Ro$ ), Froude ( $Fr$ ), and Richardson ( $Ri$ ) numbers are typically neither small nor large. In pursuit of testing this hypothesis we have revisited several classical problems, including gravitational, centrifugal/symmetric, elliptical, barotropic, and baroclinic instabilities. In all cases we find definite evidence, albeit still incompletely understood, of fluid-dynamical transitions in the neighborhood of loss of balanced integrability.

## Introduction

The general circulation of the ocean is forced by surface fluxes of heat, water, and momentum primarily at large space and long time scales. The circulation has comparably large and long scales, as well as important smaller ones associated with equatorial zonal and lateral boundary currents and with the dominant instability modes at mesoscales. All of these circulation elements approximately satisfy geostrophic, hydrostatic, and incompressible dynamical balances.

How does the energy dissipation occur for the general circulation in an equilibrium balance with the generation by surface fluxes? Some of the dissipation undoubtedly occurs within turbulent boundary layers near the surface and bottom. Some dissipation also occurs through creation of internal gravity waves by flow over

topography, with subsequent wave propagation into the interior and a wave-dynamical cascade (sometimes involving breaking) down to dissipation at small scales. Each of these routes to dissipation involves an extraction of energy from the circulation near the vertical boundaries, although the bulk of the energy resides in the vertical interior. A more local route is directly through the interior, turbulent cascade dynamics of the circulation. In oceanic general circulation models, the local route to dissipation is implied by the use of eddy diffusivities to parameterize this cascade. The purpose of this article is to examine the mechanism for the local route to dissipation, without here trying to assess the relative contributions among these alternative routes.

Our conceptual view of the mechanism is the following:

Large- and mesoscale circulations typically satisfy a balanced dynamics (as defined below), which have little interaction with the inertia-gravity wave field;

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balanced turbulent cascades are very inefficient in energy dissipation;

there are explicitly specifiable limits to the regime of balanced dynamics that are violated sometimes for the circulation;

violation of these limits leads to energy transfer to unbalanced motions;

turbulent cascades are much more efficient in their dissipation.

In this view the important bottleneck in the local route to dissipation is loss of balance and its evolutionary consequences.

Confirming or refuting this hypothesis is quite challenging, since it involves the connectedness of turbulent cascades spanning several dynamical regimes. Even diagnosis of the degree of balance can be subtle. In lieu of making a more general test of the hypothesis as yet, here we focus on the special situation of the linear instability of balanced steady currents in a rotating, stratified fluid in relation to the conditions for loss of balance. Since fluid instabilities have been the subject of much prior research, we will tell the story from both historical perspectives of early discovery and personal perspectives of the implications for the hypothesis above—skipping over most of the literature in between.

## Balanced Dynamics

The essential basis for the approximations of balanced dynamics is velocity anisotropy. In a rotating, stratified fluid with Coriolis frequency  $f$  and Brunt-Väisälä frequency  $N$  and away from boundaries, the evolution from general initial conditions or forcing by the process called geostrophic (or balanced) adjustment leads to a local anisotropy with  $u, v \gg w$ , while radiating away inertia-gravity waves. Here  $z$  and  $w$  are the coordinate and velocity components in the vertical direction, antiparallel to gravity, and  $(x, y)$  and  $(u, v)$  are their horizontal counterparts. The condition for this to occur are that the Rossby and Froude numbers,

$$Ro = V/fL \quad \text{and} \quad Fr = V/NH, \quad (1)$$

are not too large (where  $V$ ,  $H$ , and  $L$  are characteristic values for  $(u, v)$ ,  $z$ , and  $(x, y)$ ). Under these conditions the vertical momentum balance is approximately hydrostatic,

$$\phi_z \approx b, \quad (2)$$

( $\phi = p/\rho_0$  is the geopotential function,  $b = g[1 - \rho/\rho_0]$  is the buoyancy, and  $\rho_0$  is the mean density); the diver-

gence of the horizontal momentum balance is approximately

$$\nabla^2 \phi \approx \nabla \cdot f \nabla \psi + 2[\psi_{xx}\psi_{yy} - \psi_{xy}^2], \quad (3)$$

which is called gradient-wind balance ( $\nabla$  is the horizontal gradient operator); and the horizontal velocity is weakly divergent and thus can be approximately represented by a streamfunction,

$$u \approx -\psi_y \quad \text{and} \quad v \approx \psi_x. \quad (4)$$

The maximal truncation of the incompressible (Boussinesq) equations consistent with these approximations and conservation of either energy (in Cartesian coordinates,  $(x, y, z)$ ) or potential enstrophy (in isopycnal coordinates,  $(X, Y, b)$ ) is called the balance equations (Lorenz [1960]; Gent and McWilliams [1984]). (Many alternative models have been proposed for balanced dynamics; among the better ones, their similarities seem more important than their differences.) The balance equations contain no inertia-gravity wave solutions; so they are often taken as a dynamical-systems model for the (advective) slow manifold. They have second-order asymptotic accuracy as  $Ro \sim Fr \rightarrow 0$ , whereas the traditional geostrophic balance and quasi-geostrophic equations have only first-order accuracy (Gent and McWilliams [1983]). They have been shown in many analyses to accurately represent the observed state and evolution of large-scale flows in the atmosphere and ocean; for example, they often are used for initialization of weather forecasts, even ones for hurricanes. An important aspect of this accuracy is the weakness of inertia-gravity wave generation by balanced motions when  $Ro$  and  $Fr$  are not large.

The advective dynamics of balance or quasigeostrophic equations—called geostrophic turbulence—yields an inverse turbulent cascade of energy towards larger scales in  $(x, y, z)$ , hence away from dissipation by molecular viscosity at small scales, and a forward cascade of potential enstrophy (i.e., variance of potential vorticity) to its dissipation at small scales (Charney [1971]; McWilliams *et al.* [1994]). (This behavior is analogous to the turbulence in a two-dimensional fluid.) In the enstrophy inertial range,  $Ro$  and  $Fr$  are invariant as the scale decreases, at least in the limit of  $Ro, Fr \rightarrow 0$ . There is as yet much less experience with balanced turbulence at finite values of  $Ro$  and  $Fr$ , but available solutions indicate that it is only modestly more dissipative of energy (Yavneh *et al.* [1997]). It remains an open question how consistently  $Ro$  and  $Fr$  avoid increasing in the forward cascade, either in the balance equations or

more fundamental fluid dynamics: at small scales does the cascade in balanced turbulence generate inconsistencies with its justifying assumptions and how leaky is the slow manifold to unbalanced motions? Nevertheless, our present understanding is that the balance equations do not provide an efficient route to energy dissipation away from boundaries: they imply more stirring than mixing for momentum.

## Loss of Balance

An analysis for the solvability of the balance equations is made in *Yavneh et al.* [1997] and *McWilliams et al.* [1998]. To be able to integrate forward in time from a balanced state, several necessary conditions must be satisfied everywhere within the domain. Where these are violated, there is a change of type of the partial differential system and the initial- and boundary-value problem becomes ill-posed. For the balance equations in isopycnal coordinates (i.e., *Gent and McWilliams* [1984]) and  $f \geq 0$ , the conditions for loss of balance are the following:

1. Change of sign of vertical stratification,  $N^2 = \frac{\partial b}{\partial z}$ ;
2. Change of sign of absolute vorticity,  $A = f + \zeta^{(z)} = f + v_X - u_Y$  (where the horizontal derivatives denoted by capital letters are in isentropic coordinates);
3. Change of sign of  $A - |S|$  (where  $S^2 = (u_X - v_Y)^2 + (v_X + u_Y)^2$  is the variance of the strain rate).

None of these conditions occurs in the quasigeostrophic limit, since  $A, A - |S| \rightarrow f + \mathcal{O}(Ro)$  and  $N \rightarrow N_o + \mathcal{O}(Ro)$ , where  $N_o(z)$  is the resting-state stratification. Note the greater susceptibility of anticyclonic regions (i.e., with  $\zeta^{(z)}/f < 0$ ) in the second and third conditions; furthermore, note the greater susceptibility to the third condition, since  $A - |S| \leq A$ . The first and second conditions also are related to the potential vorticity,  $q = AN^2$ . Since potential vorticity is conserved on parcels, except for mixing effects, there is thus an evolutionary inhibition for an unforced flow to spontaneously develop a violation of the first and second conditions. However, there is no such constraint with respect to the third condition, which indicates another sense in which there may be a greater susceptibility to the third condition.

## Instability and Cascade

We now ask what happens when there is a loss of balance as defined above. Obviously any further integration of the balance equations is precluded. So the question must be answered in a more fundamental fluid dynamics, such as the incompressible Boussinesq equations, which have no restriction on  $Ro$  and  $Fr$  for their validity. In general, our expectation is that some inertia-gravity waves and/or more nearly isotropic turbulence will be generated—instigating some degree of enhanced dissipation—where there is a loss of balance. However, the efficiency of their generation is uncertain, as is whether the subsequent evolution systematically departs from balanced dynamics or relaxes back towards it (e.g., as a geostrophic adjustment or selective cascade and dissipation processes). In the rest of this article, we address the issue of generation and subsequent evolution only in a very particular context, viz., the linear instabilities of rotating, stratified flows which are steady, inviscid, balanced solutions of the Boussinesq equations. While this is far from the general circumstances of fluid evolution, it does provide a cleanly posed question that also is one that can be answered in part by reference to the extensive literature on fluid instabilities.

## Instability Types

Now we attempt to interpret the known instabilities in relation to the conditions for loss of balance for rotating, stratified flows where  $Ro$  and  $Fr$  are not large. It is probably unprovable that any such taxonomy of instability types can be complete and unique; however, after all the research that has gone into this topic, the landscape has become fairly well mapped.

### Quasigeostrophic Inflectional Instability

Consider the instability of a geostrophic parallel flow  $U(y, z)$  with background stratification  $N_o(z)$  and Coriolis frequency  $f(y)$  in the quasigeostrophic limit,  $Ro, Fr \rightarrow 0$ . Following *Rayleigh* [1880] and *Drazin and Howard* [1966] and ignoring vertical-boundary effects, one can derive a “Rayleigh theorem” from the potential vorticity equation that a necessary condition for inviscid instability of a non-symmetric (i.e.,  $\partial_x \neq 0$ ), normal-mode fluctuation is that the mean potential vorticity gradient,

$$N_o^{-2} Q_y = \frac{df}{dy} - U_{yy} - \left( \frac{f}{N_o} \right)^2 U_{zz}, \quad (5)$$

change sign within the domain. This is called an inflection-point instability since it involves the horizontal and/or vertical curvature of  $U$ ; depending upon which curvature is dominant, the instability is labeled barotropic or baroclinic instability. The unstable mode is itself geostrophically balanced. This is the only type of instability that occurs in the quasigeostrophic limit. It has an analytic continuation to finite  $Ro$  and  $Fr$ , where it can be expected to remain balanced over some range of these parameters. Thus, its onset conditions have nothing to do with the limits of balance, and it represents a mode of stirring within balanced dynamics.

### Gravitational Instability

The condition for the onset of gravitational instability in the limit of vanishing viscosity is  $N^2 < 0$  (Rayleigh [1916]; Chandrasekhar [1961]). This coincides with the first condition for loss of balance, and the mode of instability is unbalanced (e.g., the vertical momentum balance is non-hydrostatic).

### Symmetric Centrifugal Instability

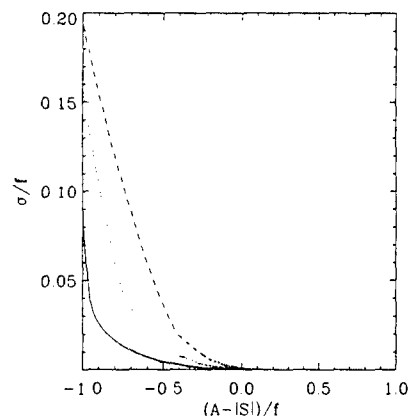
For a balanced, parallel flow  $U(y, z)$ , the necessary and sufficient condition for inviscid instability of a parallel-symmetric perturbation is a change of sign of potential vorticity  $q(y, z)$  (Hoskins [1974]). This coincides with the second condition for loss of balance. (This is a 2D problem rather than a 3D one, and McWilliams *et al.* [1998] show that the third condition is not germane in this situation.) For a balanced, axisymmetric, azimuthal flow  $U(r, z)$ , the conditions for the inviscid instability of an axisymmetric perturbation are the change of sign of either the absolute vorticity  $A$  or the absolute circulation  $C = \frac{1}{2}fr + U$  (Rayleigh [1916]; Ooyama [1966]). McWilliams *et al.* [1998] show that these coincide with the second or third conditions, respectively, for loss of balance in this case. Thus, the boundaries for onset of symmetric centrifugal instability, which has unbalanced growing modes, occur exactly at the limits of balanced evolution.

### Elliptical Instability

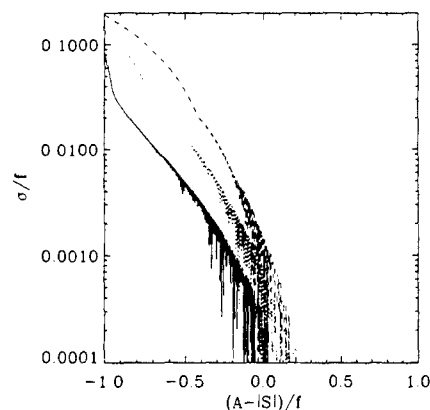
The inviscid instability of the balanced, elliptical, two-dimensional, barotropic flow in an unbounded domain,

$$\Psi(x, y) = \frac{1}{2}(\alpha x^2 + \beta y^2), \quad 1 > \frac{\beta}{\alpha} > 0, \quad (6)$$

was originally analyzed for  $f = N = 0$  (Pierrehumbert [1986]; Bayly [1986]; Craik and Criminalle [1986]) and



**Figure 1.** Growth rate,  $\sigma/f$ , for the elliptical flow (6), maximized over vertical wavenumber. Curves are shown for  $\beta/\alpha = 0.25$  (dashed), 0.11 (dotted), and 0.026 (solid). The corresponding abscissa values for the second condition for loss of balance are -0.6, -0.8, and -0.95 (see text).



**Figure 2.** As figure 1, but with logarithmic ordinate. The evident noise is because of intermittent underestimates of  $\sigma$  due to incomplete optimization searches over the very narrow unstable bands in vertical wavenumber.

later extended to  $Ro, Fr < \infty$  by Miyazaki [1993]. The unstable modes occur in bands of the vertical wavenumber. They have temporally oscillatory horizontal wavenumbers and exhibit exponential growth averaged over a wavenumber oscillation period.

The problem was revisited in *McWilliams and Yavneh* [1998] from the perspective of loss of balance: elliptical instability disappears in the quasigeostrophic limit and its onset nearly, but not precisely, coincides with the third condition for loss of balance (see Figures 1-2). The unstable mode is unbalanced. For this basic flow the vorticity and strain rate are spatially uniform and unequal in magnitude. The abscissa,  $(A - |S|)/f$ ,  $= +1$  in the quasigeostrophic limit,  $= 0$  at the third condition for loss of balance,  $= -(\alpha - \beta)(\alpha + \beta)$  at the second condition for loss of balance, and  $> +1$  for cyclonic flows. Cyclonic elliptical flows are stable for  $\alpha, \beta = O(f)$ .

### Taylor-Couette and Barotropic Instabilities

Consider Taylor-Couette flow in the gap between two axisymmetric, rotating cylinders,

$$U(r) = A r + \frac{B}{r}, \quad (7)$$

which is a viscous steady solution commonly studied in laboratory experiments. The classical instability for this flow is centrifugal (*Taylor* [1923]; *Chandrasekhar* [1961]), whose onset in the inviscid limit coincides with the second condition for loss of balance. In a neutrally stratified fluid, this is the only type of linear instability, since this profile does not have an inflection point. However, for a stably stratified fluid with small  $Fr$ , this barotropic shear flow has another class of unbalanced instabilities at finite  $Ro$  in the anticyclonic regime (*Molemaker et al.* [2001]; *Yavneh et al.* [2001]), but there is not any instability for the quasigeostrophic limit,  $Ro \rightarrow 0$ , nor for cyclonic flows with  $Ro = O(1)$ . For this new class an infinite but countable set of unstable modes exist, which differ in their cross-stream structure, each with a different narrow band of vertical wavenumbers. In Figures 3-4, the growth rates are shown for the first three modes in the thin-gap limit where  $\zeta^{(z)}$  and  $S$  are nearly uniform and equal in magnitude. As with elliptical flow, the strength of the instability strongly increases in the neighborhood of the third condition for loss of balance. (And, in this case,  $(A - |S|)/f \approx -1$  corresponds to the second condition.) The instability can be shown to involve a resonance of shear-modified neutral modes (as required in any eigenvalue problem whose eigenfrequency is either real or a

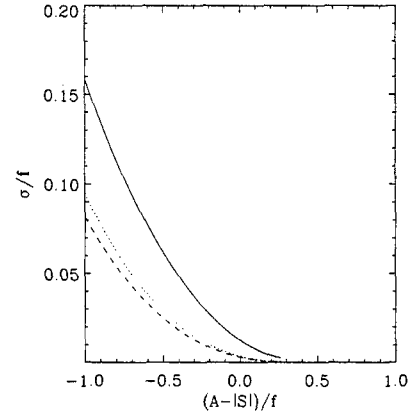


Figure 3. Growth rate,  $\sigma/f$ , for Taylor-Couette flow (7) in a narrow gap, maximized over vertical wavenumber. Curves are shown for the three gravest modes in cross-stream structure.

complex-conjugate pair); for the gravest unstable mode the resonance involves a pair of Kelvin waves propagating cyclonically along each cylinder wall, and for the other modes, one of the Kelvin modes is replaced by an inertia-gravity mode. Thus, the instability has an unbalanced dynamics. An explicit formula can be obtained for the unstable growth rate of the gravest mode,

$$\frac{\sigma}{f} \sim e^{-\gamma/Ro} \sim e^{-2\gamma/(1-X)}, \quad (8)$$

asymptotically as  $Ro \equiv \max|\zeta^{(z)}|/f \rightarrow 0^+$  and  $X \equiv (A - |S|)/f \rightarrow 1^-$  (with  $\gamma = 2$  (analytically) for the gravest mode and  $\gamma \approx 3$  (computationally) for the higher modes; n.b.,  $X$  is the abscissa in Figures 3-4). This asymptotic regime is accurately realized even in the neighborhood of the third condition for loss of balance, which implies that there is an extremely rapid weakening of  $\sigma$  with  $Ro$  and  $X$  in this neighborhood, but not an abrupt cessation at any critical value near  $X = 0$ .

Recently we have also solved the linear, inviscid stability problem for a barotropic boundary current along a straight coastline (at  $x = 0$ ),

$$V(x) = V_0 \exp(-\alpha x), \quad (9)$$

in a uniformly rotating, stratified fluid in a semi-infinite domain (i.e.,  $x \geq 0$ ). This profile also does not have an inflection point. In addition to no normal flow at the boundary, we prescribe a radiation condition at a sufficiently distant location in the interior by matching the solution to an outwardly radiating, free inertia-gravity

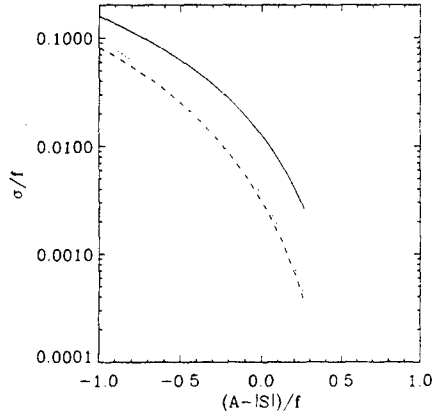


Figure 4. As figure 3, but with logarithmic ordinate

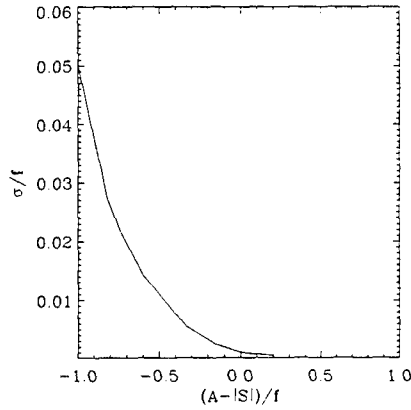


Figure 5. Growth rate,  $\sigma/f$ , for the parallel boundary current (9), maximized over along-stream and vertical wavenumbers. The abscissa value is based upon the minimum over  $x$  which occurs at  $x = 0$ .

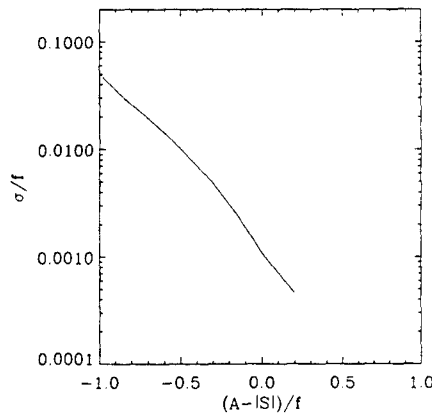


Figure 6. As figure 5, but with logarithmic ordinate

wave,

$$\frac{\partial u}{\partial x} = iku, \quad (10)$$

where  $k$  is a cross-stream wavenumber determined from vertical and along-stream wavenumbers and eigenfrequency using the dispersion relation of an inertia-gravity wave. Again there are unbalanced unstable modes for anticyclonic flows (i.e.,  $V_o/f > 0$ ) away from the quasi-geostrophic limit. In contrast with Taylor-Couette flow, with its discrete spectrum of modes in the cross-flow direction, here there is a continuous set of unstable modes for any along-stream and vertical wavenumber pair (and its corresponding cross-stream  $k$ ). In Figures 5-6, growth rates are shown for optimal along-flow and vertical wavenumbers, indicating yet again a rapid increase of the growth rate in the vicinity of the third condition for loss of balance; the functional form of  $\sigma[Ro]$  here appears to be close to that for the higher-mode, Taylor-Couette instabilities. Again, a resonance can be diagnosed with a shear-modified Kelvin mode and an inertia-gravity mode.

In summary, the three different barotropic, anticyclonic, rotating, stratified, shear flows analyzed in this and the preceding section all have an unbalanced instability whose strength rapidly increases in the vicinity of the third condition for loss of balance (in addition to the unbalanced, centrifugal instability in the vicinity of the second condition). In the hydrostatic limit, this instability is related to ageostrophic, parallel, anticyclonic flow instability in a shallow-water layer (Sato-mura [1981]; Griffiths *et al.* [1982], *et seq.*)

### Baroclinic and Kelvin-Helmholtz Instabilities

Consider a steady, spatially uniform vertical shear flow,  $U(z) \propto z$ , in a uniformly stratified and rotating fluid in a vertically bounded domain with non-isopycnal boundaries. We define the Richardson number for this flow by  $Ri = N_o^2/U_z^2$ , which is equivalent to a  $Fr^{-2}$ . The balancing buoyancy field is  $B(y, z) = N_o^2 z - fU_z y$ , and the associated quantities in the second and third conditions for loss of balance are

$$A = f \left[ 1 - \frac{1}{Ri} \right], \quad A - S = f \left[ 1 - \frac{2}{Ri} \right]. \quad (11)$$

Notice that this flow regime is anticyclonic since  $A/f < 1$ . Thus, we can make the following categorization for this flow in terms of  $Ri$  and in relation to the conditions for loss of balance:

- The quasigeostrophic limit occurs for  $Ri \rightarrow \infty$ . It has a geostrophic baroclinic instability (Charney

[1947]; *Eady* [1949]), whose "inflection point" in this particular case occurs at the vertical boundaries (see above).

- The third condition is satisfied if  $Ri < 2$ .
- The second condition is satisfied if  $Ri < 1$ .
- The classical (non-rotating) condition for the onset of Kelvin-Helmholtz instability (*Miles* [1961]; *Howard* [1961]) is satisfied if  $Ri < \frac{1}{4}$ .
- The first condition is satisfied if  $Ri < 0$ ; this is gravitational instability (see above).

Analyses of this problem in *Stone*, [1966,1970] and *Nakamura* [1988] show that a centrifugal instability does occur near  $Ri = 1$  for both zonally symmetric and asymmetric fluctuations and that another ageostrophic instability (with shorter zonal wavelengths than the geostrophic instability) occurs for even larger values of  $Ri$ , though its onset value is not well determined. This latter instability is shown in Figure 2 of *Stone* [1970], with an accompanying remark that it has "growth rates  $Im\sigma$  which decrease as  $Ri$  increases, and may in fact still be unstable for  $Ri \geq 2$ , but if so the growth rates were too small to be found by our numerical method, because of the near singular behavior of the coefficients of the differential equation when  $Im\sigma$  is very small." Confirmation is presented in Figure 8 of *Nakamura* [1988] which shows that  $Im\sigma$  is strongly decreasing as  $Ri$  increases; it is somewhat unclear exactly how large a  $Ri$  value he obtained unstable solutions for, but his figure suggests it is at least as large as  $Ri = 2$ . An interpretation of the shortwave instability is presented (p. 3261): it involves a resonance between a boundary-trapped shear mode and an inertia-gravity mode, with an "inertia critical level" that limits the vertical extent of the eigenmode. Thus, although further examination is needed of the behavior of  $Im\sigma$  in the vicinity of the third condition for loss of balance, the instabilities for this flow appear to support quite well the hypothesis we have advanced.

## The Rest of the Quest

We have presented here a survey of the known instabilities of rotating, stably stratified fluids. Barotropic and baroclinic inflectional instabilities are the only types that occur in the quasigeostrophic limit, and several others arise in the neighborhood of the limits of balance as determined from the balance equations. These latter include gravitational, symmetric centrifugal, and elliptical, as well as presently unnamed variants

of barotropic and baroclinic instabilities. The instability onset is generally understood to be sharp with respect to the  $N^2$  and  $A$  conditions for loss of balance, which correspond to the gravitational and centrifugal instabilities. We have shown for the elliptical, Taylor-Couette, and barotropic boundary-current instabilities that the onset behaviors are smooth in the transition across the  $A - |S|$  condition, though still quite steep, with exponential or steeper dependences for the growth rate,  $\sigma(A - |S|)$ . The onset behavior near the  $A - |S|$  condition has yet to be as well determined in the ageostrophic baroclinic instability problem, but previous studies indicate it is also steep. Kelvin-Helmholtz instability does not fit in this categorization scheme because its onset occurs well beyond the  $A$  and  $A - |S|$  conditions.

Thus, we conclude that these problems give confirmation, *pro tem*, of the hypothesis that the limits of balance are indicative of transition from the larger-scale regime of inverse energy cascades in anisotropic flows to the smaller-scale regimes of forward energy cascade to dissipation of more nearly isotropic flows and intermittently breaking inertia-gravity waves. There is still need for refining our understanding of  $\sigma(Ri, Ro, Fr)$  in the neighborhood of the conditions for loss of balance, especially for the less familiar third condition. Now that we know that pure barotropic and baroclinic instabilities fit the hypothesis, it would be worthwhile to further test it with more general shear profiles (e.g., the "coastal"  $V(x, z)$  profile that *Barth* [1994] has shown to have an ageostrophic baroclinic instability). Beyond these linear instability problems lies the challenge of aptly diagnosing flow evolution near the loss of balance conditions in general nonlinear initial- and boundary-value problems. Nevertheless, we interpret the results in hand as indicating that the loss-of-balance transitions are steep but fuzzy, consistent with the view that the balanced slow manifold is itself modestly fuzzy and leaky to more efficiently cascading, unbalanced motions.

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